Size of quantum networks

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The metric structure of bosonic scale-free networks and fermionic Cayley-tree networks is analyzed focousing on the directed distance of nodes from the origin. The topology of the netwoks strongly depends on the dynamical parameter T, called temperature. At $T=\infty$ we show analytically that the two networks have a similar behavior: the distance of a generic node from the origin of the network scales as the logarithm of the number of nodes in the network. At $T=0$ the two networks have an opposite behavior: the bosonic network remains very clusterized (the distance from the origin remains constant as the network increases the number of nodes) while the fermionic network grows following a single branch of the tree and the distance from the origin grows as a power-law of the number of nodes in the network.

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Complex networks representing systems of interacting units have been studied and classified [1,2] according to their different geometrical and topological properties. In between complex networks scale-free networks,with power-law connectivity distribution, have been found to describe different systems of nature and society [3–5]. Recently several models [3–5] have been formulated that generate such structures, the prototype of them being the BA model [6]. Scale-free networks are particulary interesting because their highly inhomogeneous structure induces peculiar effects in the dynamical models that can be defined on them, as the absence of percolation [7] and epidemic threshold [8] or an infinite Curie temperature for the paramagnetic to ferromagnetic transition in the frame of the Ising model [9–12]. Besides, the investigation of the metric stucture [13–16] of these networks is of large interest. It was first empirically found [13] and then analytically derived [14,15] that scale-free networks are characterized by having a mean distance $<d>$ between nodes scaling like the logarithm of the system size N, $<d>\sim \log(N)$.

The similarities between the structure of a BA network and a traditional Cayley tree have been recently studied. It has been found that they share high similarities. In fact they are both generated by the subsequent addition of a same elementary unit (a node connected to m links) attached to the rest of the network in the direct or in the reversed direction [17]. The symmetry between these two types of networks is evident if we assume that each node i has an innate quality or “energy” ϵ_i and that the dynamics is parametrized by a variable T called temperature. In this case it is possible to observe that the BA model becomes a limiting case of a scale-free network described by a Bose distribution of the energies to which the incoming links point [18], while the growing Cayley tree network is described by a Fermi distribution of the energies at the interface [19]. Quantum networks (the bosonic scale-free network and the fermionic Cayley tree network) evolve around a well defined core of initial nodes.

In this paper we focus our attention on the metric structure of quantum networks and in particular on their size, i.e. we measure the distance of a generic node i from the origin of the network and its dependence on the time t_i in which the node i has been added to the network. We derive the expression for the average distance $<\ell(t_i)>$ of node i introduced at time t_i from the origin, in quantum networks at $T=\infty$. We find in agreement with [13–15] that $<\ell(t_i)>\sim \log(t_i)$ in the bosonic scale-free network and we show for $T=\infty$ a similar behavior in fermionic networks.

At different values of T the behavior of $<\ell(t_i)>$ changes drastically. The distance of node i from the origin depends less strongly on the time t_i of its arrival as the temperature decreases. At sufficiently low temperature $\ell(t_i)$ remains constant as a function of t_i. On the contrary in the fermionic network, at $T=0$, when the dynamics becomes extremal, the network evolves far away from the origin and the distance of a node i from the origin of the network grows as a power-law of the time t_i of its arrival in the network.

A. Distances from the origin in the bosonic network

The bosonic network is a generalization of the well known BA network [6]. In this model, a new node with m links is added to the network at each time step. Each node i has an innate quality or 'energy' ϵ_i extracted form a probability distribution $p(\epsilon)$. The way the new links are attached follows a generalized preferential attachment rule: the probability Π_j that an existing node j acquires a new link depends both on its connectivity $k_j(t)$ and its energy ϵ_j, i.e.

$$\Pi_j = \frac{e^{-\beta\epsilon_j} k_j(t)}{\sum_s e^{-\beta\epsilon_s} k_s(t)}. \tag{1}$$

with the parameter $\beta = 1/T$ tuning the relevance of the energy ϵ_j with respect to the connectivity $k_j(t)$. This network displays a power law connectivity distribution $P(k)\sim k^{-\gamma}$ with $\gamma\in[2,3]$ depending on the $p(\epsilon)$ distribution and the inverse temperature β [18]. In the $T=\infty$ limit ($\beta=0$) the network reduces to a scale-free BA network [6] with an average connectivity k_i of node i that grows in time as a power-law with exponent 1/2

$$k_i(t) = m\sqrt{\frac{t}{t_i}}, \tag{2}$$

where t_i is the time in which node i was added to the network. The probability $p_{i,j}$ that two nodes i and j are connected by a link is given by Eq. (1). At $T = \infty$ ($\beta = 0$), taking into account that at each time m new links are added to the network, the probability p_{t_i,t_j} is given by m times Eq. (1). After substituting (2) into Eq. (1) we obtain,

$$p_{t_i,t_j} = \frac{m}{2}\frac{1}{\sqrt{t_i t_j}}. \tag{3}$$

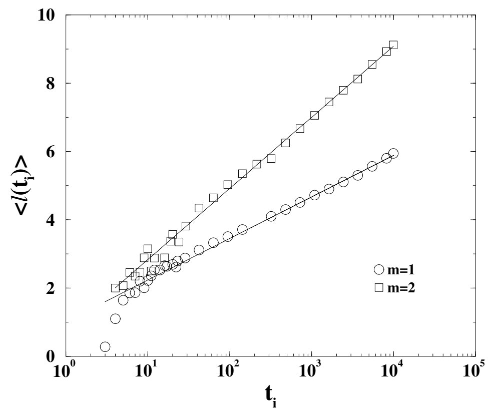

FIG. 1. Mean distance from the origin $< \ell(t_i) >$ of the nodes arrived at time t_i in a BA network (a bosonic network at $T = \infty$) with $m = 1, 2$. The solid lines indicate the theoretical prediction Eq. (6).

The number of directed paths of lenght ℓ connecting node i, introduced in the network at time t_i, to node i_0 belonging to the original core of the network ($t_{i_0} = 1$) is given by the mean value of the number of paths connecting node i to i_0 and passing through the points $i_0, i_1, i_2, \ldots i_{\ell-1}, i$ with $t_{n+1} > t_n$. Indicating each node with the time of its arrival in the network, and replacing the sum over the nodes with the integrals over t_n, we obtain for $n_\ell(t_i)$,

$$n_\ell(t_i) = \int_1^{t_f} dt_1 p_{t_{i_0},t_1} \int_{t_1}^{t_f} dt_2 p_{t_1,t_2} \ldots \int_{t_{\ell-2}}^{t_f} dt_{\ell-1} p_{t_{\ell-1},t_i}. \tag{4}$$

Using (3) valid in the $T = \infty$ limit, we obtain

$$\begin{aligned} n_\ell(t_i) &= \left(\frac{m}{2}\right)^\ell \int_{t_{i_0}}^{t_i} dt_1 \int_{t_1}^{t_i} dt_2 \ldots \\ &\quad \ldots \int_{t_{\ell-2}}^{t_i} dt_{\ell-1} \frac{1}{t_1}\frac{1}{t_2}\cdots\frac{1}{t_{\ell-1}}\frac{1}{\sqrt{t_{i_0} t_i}} \\ &= \frac{1}{(\ell-1)!}\left(\frac{m}{2}\log\left(\frac{t_i}{t_{i_0}}\right)\right)^{\ell-1}\frac{m}{2\sqrt{t_{i_0} t_i}}. \end{aligned} \tag{5}$$

This means that the mean distance between node i and a node i_0, such that $t_{i_0} = 1$, calculated only on directed paths, follows a Poisson distribution with average size

$$< \ell(t_i) >= \frac{m}{2}\log(t_i). \tag{6}$$

The distribution of the number of directed paths of lenght ℓ starting from the origin of the network is proportional to the integral of $n_\ell(t_i)$ over t_i

$$P(\ell) \propto \int_1^t n_\ell(t')dt' = (-m)^\ell \left(1 - \frac{\Gamma(\ell, \log(t)/2)}{\Gamma(\ell)}\right). \tag{7}$$

In Fig. 2 we show the agreement between the numerical results and Eq. (7) for a bosonic network of 10^4 nodes at $T = \infty$ and $m = 1, 2$.

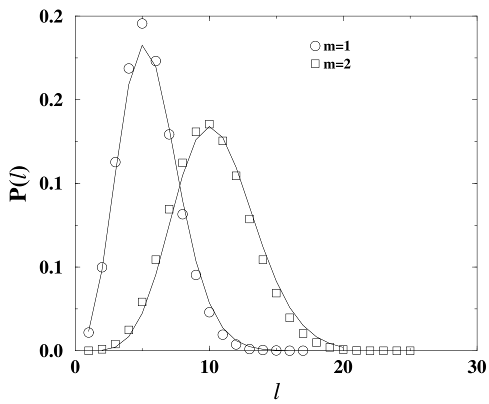

FIG. 2. Distribution of the number of directed paths of lenght ℓ in a BA network (a bosonic network at $T = \infty$) for $m = 1, 2$. The solid lines are the analytic predictions (7).

The topology of the bosonic network changes as a function of the inverse temperature β. In particular for a distribution $p(\epsilon)$ such that $p(\epsilon) \to 0$ as $\epsilon \to 0$ we know [18] that there is a critical temperature T_c below which the network has a topological transition and its structure is dominated by a single node that grabs a finite fraction of all the links. For $T > T_c$ ($\beta < \beta_c$) the network is in the so-called "fit-gets-rich" (FGR phase) while for $T < T_c$ ($\beta > \beta_c$) the network is in the so-called Bose-Einstein condensate phase (BE phase). To visualize this transition we have plotted the bosonic network with an energy distribution

$$p(\epsilon) = \frac{1}{\theta+1}\epsilon^\theta, \ \epsilon \in (0, 1) \tag{8}$$

where $\theta = 0.5$, $m = 1$, above (Fig. 3a) and below (Fig. 3b) the phase transition. The network has been designed in order to underline the hierarchical structure of the network. Starting from the single node at the origin of the tree, we have placed all the nodes that are directly attached to it on a semicircle of unitary radius, each node i separated from the next one by the angle $\Delta\alpha_i$ proportional to its connectivity,i.e.

$$\Delta\alpha_i = \frac{k_i}{\sum_{j\in N(i)} k_j} \tag{9}$$

where $N(i)$ are the nearest neighbors of node i added at a time $t_j > t_i$. We have repeated the same construction for all the nodes of the network in such a way that all the nearest neighbors of node i are on a semicircle of radius r_i with

$$r_i = r_k \frac{k_i}{\sum_{j\in N(i)} k_j}, \tag{10}$$

where k is the node to which the node i has been attached at time t_i. From Fig. 3 it is clear the change in the topology of the network at the critical temperature with the emergence of a single node that grabs a finite fraction of all the links in the Bose-Einstein condensate phase.

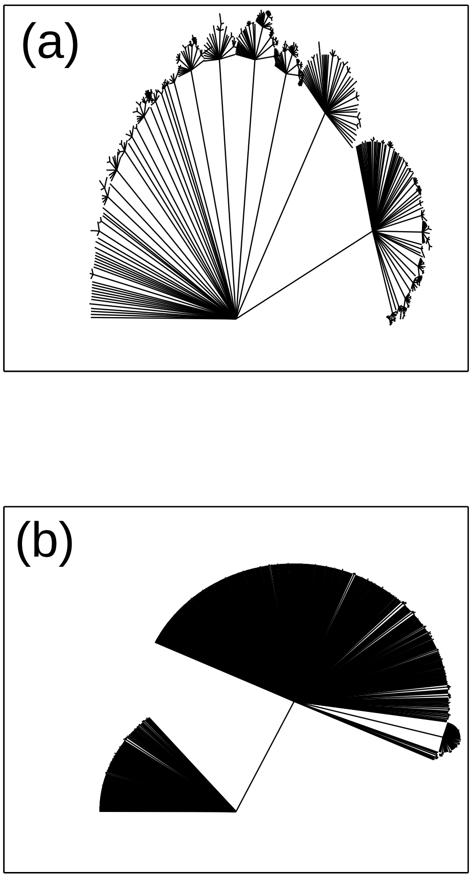

FIG. 3. Graphic representation of the bosonic network. Graph (a) represents a network with $m = 1$, energy distribution given by Eq. (8) where $\theta = 0.5$ at $\beta = 0$ (the BA network) and graph (b) represents a network with the same parameters m and θ but with $\beta = 3.0$ (Bose-Einstein condensate network).Thenumber of nodes both networks is $N = 10^3$.

As the topology of the network changes, the behavior of $\ell(i)$ as a function of t_i changes too. In fact we have

$$< \ell(i) >= a(\beta)\log(t_i) \tag{11}$$

where the coefficient $a(\beta)$ is a decreasing function of the inverse temperature $\beta = 1/T$. In Fig. 4 we report $< \ell(i) >$ for a bosonic network with $p(\epsilon)$ of the type (8) with $\theta = 0.5$ at different values of the inverse temperature β, above and below the critical value $\beta_c \sim 1/0.6 \sim 1.7$ [18]. The coefficient $a(\beta)$ decreases with the inverse temperature β and saturates to the zero value $a(\beta) \sim 0$ at a temperature T' lower than the critical temperature $T' < T_c$. In Fig. 4 we show this for a network of size $N = 10^4$ and $\theta = 0.5$ with $a(\beta) \simeq 0$ for $\beta > \beta' = 3.0 > \beta_c \sim 1.7$ [18].

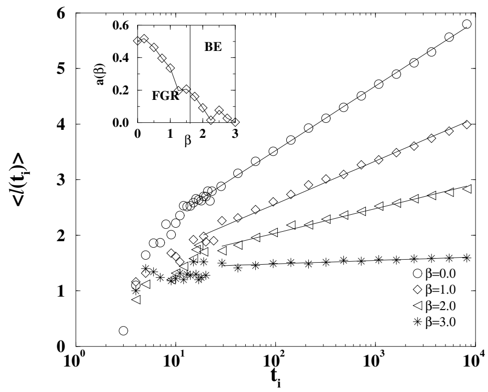

FIG. 4. Distances form the origin in a bosonic network with $m = 1$ and $\theta = 0.5$ as a function of β. In the Inset we show the value of the coefficient $a(\beta)$ defined in (11) as a function of β that saturates to zero for $\beta > \beta' = 3.0$

B. Distances from the origin in the fermionic network

The fermionic network [19]. is a growing Cayley tree, where the innate qualities of the nodes (energies) define their different branching tendency. Starting at time $t = 1$ from a node i_0 at the origin of the network, the node i_0 at time $t = 2$ grows and m new nodes are directly connected to it. Each node i has an energy ϵ_i extracted from a given $p(\epsilon)$ distribution. At each timestep a new node with connectivity one (at the interface) is chosen to branch, giving rise to m new nodes.

We assume that nodes with higher energy are more likely to grow than lower energy ones. In particular we take Π_i, the probability that a node i of the interface (with energy ϵ_i) grows at time t, to be

$$\Pi_i = \frac{e^{\beta\epsilon_i}}{\sum_{j\in Int(t)} e^{\beta\epsilon_j}}. \tag{12}$$

where the sum in the denominator is extended to all nodes j at the interface $Int(t)$ at time t. The model depends on the inverse temperature $\beta = 1/T$. In the $\beta \to 0$ limit, high and low energy nodes are equally probable to grow and the model reduces to the *Eden model*. In the $\beta \to \infty$ limit the dynamics becomes extremal and only the nodes with the highest energy value are allowed to grow.In this case the model reduces to *invasion percolation* [20,21] on a Cayley tree.

Let us assume that $\beta = 0$ ($T = \infty$). The probability Π_i that a node i of the interface $Int(t)$ grows at time t is given by

$$\Pi_i = \frac{1}{N_{Int}(t)}, \tag{13}$$

where $N_{Int}(t)$ is the total number of active nodes. Since at each timestep a node of the interface branches, and m new nodes are generated, after t timesteps the model generates an interface of $N_{Int}(t)$ nodes, with

$$N_{Int}(t) = (m-1)t + 1. \tag{14}$$

Lets denote by $\rho(t, t_i)$ the probability that a node, born at time t_i is still at the interface at time t. Since every node i of the network grows with probability Π_i Eq. (13) if it is at the interface, in mean field $\rho(t, t_i)$ follows

$$\frac{\partial \rho(t, t_i)}{\partial t} = -\frac{\rho(t, t_i)}{N_{Int}(t)}. \tag{15}$$

Replacing (14) in (15) in the limit $t \to \infty$ we get the solution

$$\rho(t, t_i) = \left(\frac{t_i}{t}\right)^{1/(m-1)}. \tag{16}$$

Consequently each node i that arrives at the interface at time t_i, remains at the interface with a probability that decreases in time as a power-law. The probability $p_{i,j}$ that a node i is attached to a node j (arrived in the network at a later time $t_j > t_i$) is given by the r. h. s. of Eq. (13) calculated at time t_j. Taking into account Eq.(16) and the rate m of the addition of new nodes, we obtain for p_{t_i,t_j}

$$p_{t_i,t_j} = \frac{m}{(m-1)t_j}\left(\frac{t_i}{t_j}\right)^{1/(m-1)}. \tag{17}$$

The number $n_\ell(t_i)$ of paths of lenght ℓ that connect a node i, introduced at time t_i, to the origin i_0 is given by the average number of paths connecting a node i to a node i_0 and passing through the points $i_0, i_1 i_2, \dots i_{\ell-1}, i$ with $t_{n+1} > t_n$. Indicating each node with the time of its arrival in the network and the sum over the nodes with the integrals over t_n, we obtain for $n_\ell(t_i)$,

$$n_\ell(t_i) = \int_{t_{i_0}}^{t_i} dt_1 p_{t_{i_0},t_1} \int_{t_1}^{t_i} dt_2 p_{t_1,t_2} \dots \int_{t_{\ell-2}}^{t_i} dt_{\ell-1} p_{t_{\ell-1},t_i} \tag{18}$$

and, using (17) we obtain

$$\begin{aligned} n_\ell(t_i) &= \left(\frac{m}{m-1}\right)^\ell \int_{t_{i_0}}^{t_i} dt_1 \int_{t_1}^{t_i} dt_2 \dots \\ &\dots \int_{t_{\ell-2}}^{t_f} dt_{\ell-1} \frac{1}{t_1}\frac{1}{t_2}\dots\frac{1}{t_{\ell-1}}\left(\frac{t_{i_0}}{t_i}\right)^{1/(m-1)} \\ &= \frac{1}{(\ell-1)!}\left(\frac{m}{m-1}\log\left(\frac{t_i}{t_{i_0}}\right)\right)^{\ell-1}\left(\frac{t_{i_0}}{t_i}\right)^{1/(m-1)}. \end{aligned} \tag{19}$$

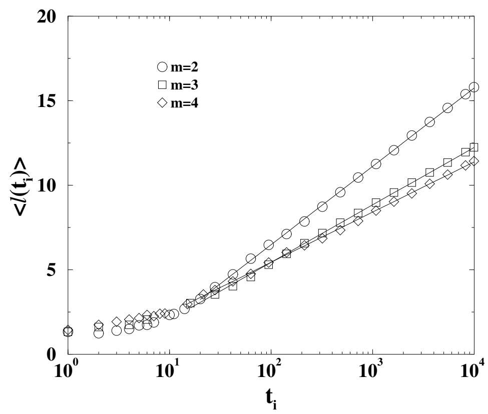

FIG. 5. Distance from the origin in a fermionic network at $T = \infty$ ($\beta = 0$) for networks of $N = 10^4$ nodes with $p(\epsilon)$ uniform between zero and one and with $m = 2, 3, 4$. The solid lines are the theoretical predictions Eq.(20).

This means that the mean distance between a node t_i and the origin follows a Poisson distribution with average size

$$< \ell(t_i) >= \frac{m}{m-1}\log(t_i). \tag{20}$$

As in the bosonic network, in the fermionic network at infinite temperature ($\beta = 0$) the distance $< \ell(t_i) >$ of node i from the origin grows logarithmically with the time t_i.

In Fig. 5 we report the analytical simulation of a fermionic network with $p(\epsilon)$ uniform between zero and one, at $T = \infty$ and $m = 2, 3, 4$.

As the temperature decreases, the topology of the network changes drastically, in Fig. 6 we show the Cayley tree with $p(\epsilon) = 1$, $\epsilon \in (0, 1)$ and $m = 2$ at infinite temperature ($\beta = 0.0$) and at low temperature ($\beta = 20.0$). At high temperature the network grows homogeneously in each direction while at low temperature it evolves following only a single branch of the tree.

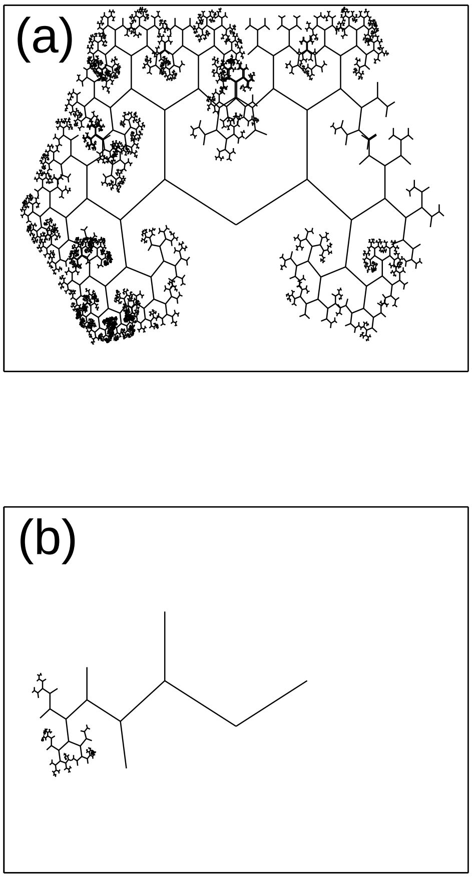

FIG. 6. The fermionic network with $m = 2$ at $T = \infty$ (Graph (a)) and at temperature $T = 0.05$ (Graph (b)).The number of nodes in both networks are $N = 10^4$.

The distance of a node i from the origin of the networks grows logarithmically with t_i at $T = \infty$ ($\beta = 0$). As the temperature decreases the behavior of $< \ell(t_i) >$ gets steeper. For $p(\epsilon)$ uniformly distributed between zero and one, in the extremal case $T = 0$ ($\beta = \infty$) when the node of highest energy grows deterministically at each time step, we have a dramatic change in the behavior and $< \ell(i) >$ grows as a power-law of t_i,

$$< \ell(i) > \propto (t_i)^{\zeta} \tag{21}$$

$\zeta = 0.55 \pm 0.05$ from the numerical results reported in Fig. 7.

In conclusion we have shown that bosonic and fermionic network are not only simmetrically built [17] but also at $T = \infty$ they are characterized by a distance $< \ell(t_i) >$ from the origin that grows like the logarithm of of the time t_i. On the contrary, in the limit $T = 0$ they behave in a opposite way: the bosonic network stays highly clusterized with a distance from the origin that remains constant as the network evolves, in the fermionic network the distance $< \ell(t_i) >$ grows like a power-law of the time t_i.

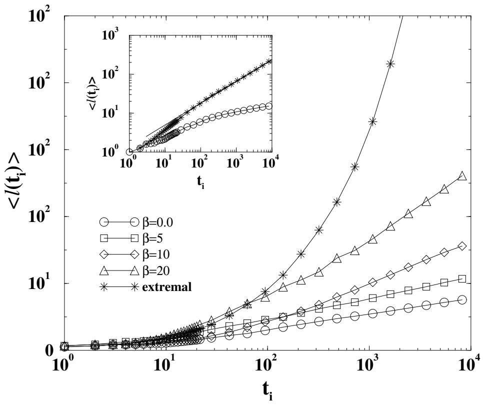

FIG. 7. Distance from the origin in a femionic network with $m = 2$ and $p(\epsilon)$ uniformly distributed between zero and one at different temperatures. At $\beta = 0$ ($T = \infty$) we have the predicted logarithmic behavior Eq. (20) while in the extremal case $\beta = \infty$ ($T = 0$) the network grows as a power-law of the network size. The solid line in the Inset is the power-law fit Eq. (21) with $\zeta = 0.55 \pm 0.05$.

I. ACKNOWLEDGMENTS

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