Efficiency of informational transfer in regular and complex networks

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We analyze the process of informational exchange through complex networks by measuring network efficiencies. Aiming to study nonclustered systems, we propose a modification of this measure on the local level. We apply this method to an extension of the class of small worlds that includes *declustered* networks and show that they are locally quite efficient, although their clustering coefficient is practically zero. Unweighted systems with small-world and scale-free topologies are shown to be both globally and locally efficient. Our method is also applied to characterize weighted networks. In particular we examine the properties of underground transportation systems of Madrid and Barcelona and reinterpret the results obtained for the Boston subway network.

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I. INTRODUCTION

Modeling of complex systems as networks of coupled elements, such as chemical systems [1,2], neural networks [3], epidemiological [4,5], and social networks [6] or the Internet [7], has been a subject of intense study in the last decade. Networks can be classified into three broad groups: (i) regular networks, (ii) random networks, and (iii) systems of complex topology, including small-world [8,9] and scale-free networks [6,10–13]. In addition, networks can be unweighted or weighted, depending on whether links are equal or different. Weights can be physical distances, times of propagation of informational packets, inverse velocity of chemical reactions, strength of interactions, etc. [14–17].

Commonly used regular networks are square or cubic lattices, both having squares as basic cycles [17–19]. Aiming to describe clustering in social networks—i.e., to account for triangles of connected nodes as basic cycles [20]—clustered rings were introduced, in which each site was linked to all its neighbors from the first up to the Kth [4,14,21,22]. The study of random graphs [23–26] was motivated by the observation of real networks that often appeared to be random. Complex networks having a topology in between those of random and regular networks were later introduced. An outstanding example is the small-world model [8,27]. Small worlds are constructed by randomly rewiring links of a regular graph [8] (so that the number of links remains constant, while the structure is changed) or adding new links to it [28] (changing both the structure and the number of links) with a probability p. In this way, shortcuts between distant nodes are created. The rewiring and adding probability p indicates, on average, the degree of disorder of the network (it varies from $p=0$ for a regular up to $p=1$ for a random graph). Small worlds are highly clustered, showing triangles of nodes like regular networks, while having small distances between sites as in random systems [4,5,29–31]. Recently, it was realized that many social and biological networks had a degree (connectivity) distribution that was not Poisson like, as in random and small-world networks, but rather a power law. Such systems were called scale-free systems [10–13] and are continuously growing open systems constructed by attaching new nodes preferentially to nodes of higher degree [11]. Various modifications of this basic procedure have been proposed: nonlinear preferential attachment [32], initial attractiveness [33], and aging of sites and degree constraints [13] or node fitness [34]. Moreover, introducing a finite memory of the nodes, large highly clustered systems can be obtained, representing a combination of scale-free networks and regular lattices [35].

Our aim is to compare the efficiency of informational transfer on the regular and complex networks described above. In Sec. II we describe the networks and define the quantities used to characterize them. In Sec. II A an extension of the class of small worlds, referred to as *declustered*, is proposed. In Sec. II C we discuss the efficiency measures reported in the literature and propose the alternatives required to handle nonclustered systems. Some of the new measures defined here are an extension of those reported in [36]. Section III is devoted to discuss the properties of various unweighted networks. Introducing physical distances, efficiencies of weighted networks are defined in Sec. IV and used to examine underground transportation systems. Our achievements are summarized in Sec. V.

II. METHODS

A. Types of networks

The networks analyzed here are clusters of the square lattice, clustered and declustered regular rings, as examples of regular systems, and clustered Watts-Strogatz and declustered small worlds, and ordinary Albert-Barabási scale-free networks, representing complex systems. All networks are chosen so that the ratio between the number of links, N_l, and the number of sites, N, is kept constant $N_l/N=2$ (this gives an average connectivity $\langle k\rangle=4$).

Concerning regular two-dimensional (2D) networks, calculations were performed for $l\times l$ clusters of the square lattice, with periodic boundary conditions: $\text{node}(i+l,j)=\text{node}(i,j)$ and $\text{node}(i,j+l)=\text{node}(i,j)$. In the case of regular rings, we analyze the simplest clustered lattices with ad-

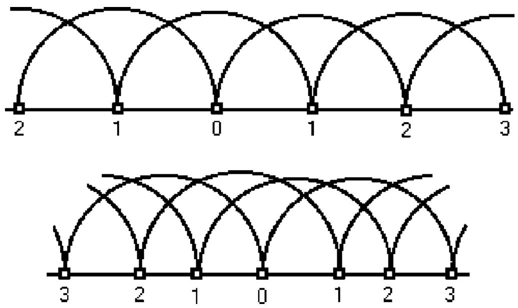

FIG. 1. Illustrates the link structure in clustered rings (upper) with connections to the second nearest neighbors ($K=2$) and de-clustered rings (lower) with connections to the third nearest neighbors ($K=\overline{3}$).

ditional connections only to the next-nearest neighbors ($K=2$) [29,37]; see Fig. 1. In addition, we study rings with a zero clustering coefficient constructed by adding links from each site to *only* its nth neighbors. We call them *declustered regular ring networks* and designate their coordination parameter as $K=\overline{n}$ (shown also in Fig. 1). Therefore, in our notation $K=n$ means that each site is additionally linked to *all* of its ring neighbors from the second to the nth, while $K=\overline{n}$ implies that only links to its nth neighbors are added. For such declustered networks, basic loops are squares for any $\overline{n}$, with edges on sites i, $i+n$, $i+1$, and $i+n+1$. Our motivation to analyze networks with a negligible clustering comes from the fact that such systems can be quite often found in nature or artifacts (for instance, in transportation underground networks). Such networks are usually very sparse with $N_l \approx N$ [36].

We differentiate between ordinary clustered small worlds and *declustered* small worlds, depending on the initial regular network. We will construct small-world networks starting from clustered and declustered regular networks with $K=2$ and $K=\overline{3}$, respectively. Moreover, as our focus is on the effects of network topology, we compare networks with the same links-to-size ratio. Thus, shortcuts are created by randomly *rewiring* links between each site and its more distant neighbors with probability $2p \leqslant 1$, while connections to the nearest neighbors are kept unchanged. In this way, the ring structure is preserved and the problem of disconnected graphs is avoided [28]. The total number of rewired links would approach $pN \leqslant N/2$ for large N. Finally, we construct scale-free networks starting with a fully connected graph of $m_0=5$ nodes and $n_0=10$ links. At each step a new node is added, with $m=2$ edges to the old nodes, so that the ratio $N_l/N=2$ is kept constant.

B. Average path length and clustering coefficient

The structural properties of a graph are usually quantified by the average path length L and the clustering coefficient C [8,13]. The average path length is calculated as the network average of the shortest graph distances between two nodes (d_{ij}) for all possible pairs:

$$L = \frac{1}{N(N-1)} \sum_{i \neq j} d_{ij}, \quad (1)$$

defined for connected graphs for which all d_{ij} are finite.

The clustering coefficient C measures to which extent are neighbors of each site connected to each other. It is calculated as a network average:

$$C = \frac{1}{N} \sum_{i} C_i, \quad (2)$$

where C_i is the ratio of the existing number of links between the neighbors of a site i and the maximum possible number of them $k_i[k_i-1]/2$, k_i being its connectivity (degree).

It is worth noting that, although the clustering coefficient of almost all real networks is very high [13], it seems that it is much less important for the collective dynamical behavior of a network than the average path length. Moreover, in some cases, like square lattices or declustered ring networks, the usual clustering coefficient fails to correctly quantify the underlying order of the hierarchical structure of the system. Recently it was proposed that such gridlike structures should be characterized by a grid coefficient, numbering the fraction of all the loops of length 4 (*quadrilaterals*) passing through each node [38,39]. Analysis of real networks, such as Internet, Web, and scientific coauthorship, reveals a good local rectangular clustering [38]. However, similarly to ordinary clustering coefficient based exclusively on triangles, this new measure concentrates only on square loops. Any attempt to analyze more sparse networks with longer basic cycles would call for the introduction of new coefficients of even higher orders. It would be much more useful to find a single measure of local properties that could be applied to any type of networks. Furthermore, it is not clear what is the physical meaning of these various coefficients and how would they be related to the dynamical behavior of the network.

C. Efficiency of informational exchange

Another approach to analyze global and local properties of a network is introducing the concept of efficiency of informational exchange through the network [36,40,41].

1. Global efficiency

We assume that it is easier to transfer information from one site to another if they are closer to each other. Therefore, the efficiency in the communication between two sites i and j is calculated as the inverse of the shortest path length d_{ij} between these two sites: $\epsilon_{ij}=1/d_{ij}$. Contrary to the average path length, efficiency can be determined even if there is no path between i and j, as in the case of disconnected graphs: $\lim_{d_{ij}\to\infty} \epsilon_{ij}=0$. The global efficiency of the network is calculated as the average over all pairs of nodes [36],

$$E^g = \frac{1}{N(N-1)} \sum_{i \neq j} \frac{1}{d_{ij}}, \quad (3)$$

and is normalized to its possible largest value $N(N-1)$, for totally connected graphs having $N(N-1)/2$ edges. Physically, E^g measures the efficiency of a system with parallel exchange of information, while $1/L$ accounts for the efficiency of a sequential propagation of a single informational packet along the network. In the case of real networks, E^g

gives a better measure for the transfer of information than $1/L$, although quite often $1/L$ could be a reasonable approximation of E^g [36].

2. Local efficiency

A similar definition can be implemented on a local level. As a counterpart of the clustering coefficient C, the local efficiency could be defined as an average efficiency of the local subgraphs of the first neighbors $(j,k \in \Gamma_i)$ of each site i [36]:

$$E_0^l = \frac{1}{N}\sum_i \frac{1}{k_i[k_i-1]}\sum_{j\neq k\in\Gamma_1}\frac{1}{d^0_{jk/i}}. \qquad (4)$$

Here $d^0_{jk/i}$ is the shortest path length between sites j and k passing only through other elements of that local subgraph of neighbors (Γ_1), which is indicated by the superscript 0. In such a way, the clustering coefficient is equal to the local efficiency when only direct connections between j and k are considered.

We propose a new definition of local efficiency, taking into account that neighbors of each reference site i can actually exchange information along paths including sites which do not necessarily belong to the local subgraph of i's neighbors $(m \notin \Gamma_1)$. In order to measure the efficiency of communication between the nearest neighbors of i when it is removed, we must only exclude site i from such a path $(d_{jk/i})$:

$$E_1^l = \frac{1}{N}\sum_i \frac{1}{k_i[k_i-1]}\sum_{j\neq k\in\Gamma_1}\frac{1}{d_{jk/i}}. \qquad (5)$$

When applying such a concept on graphs without triangle cycles, we will see that they can transfer the information quite efficiently on a local level, although their clustering coefficient is zero. It is worth noting that in the definition of [36] [see Eq. (5)] local efficiency depends only on the links present in the graph Γ_1 of the first neighbors of site i. It is calculated excluding both site i and the rest of the network $(m \notin \Gamma_1)$. In the new definition, however, local efficiency depends on the full network topology and is calculated cutting off only site i.

The clustering coefficient was introduced to measure the closeness of sites or the locality of a network [8]. Locality tells us up to what extent the neighbors of a site remain close to each other after this site is cut off. Regular networks are precisely those which show the highest locality. The criteria used for the calculation of the clustering coefficient take direct connections as a substitute for closeness. From our standpoint, it is not necessary to have two sites directly connected in order to conclude that they are close to each other. They will be far away only if the length of the path that connects them (going through the rest of the network, except site i) turns out to be large. In this way, the level of closeness (or locality) among neighbors of i depends on network topology. At first sight, our measure of locality mixes global and local properties. However, we must note that the path between sites that are close to each other does not go throughout the whole remaining network, but only through the close surroundings of these sites. The immediate surroundings of these sites will overlap in a great extent, defining a common local region. Therefore, our definition of local efficiency is logically consistent, as it depends mainly on local topology.

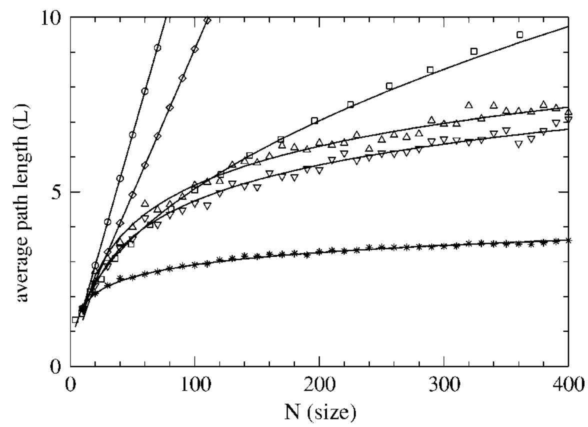

FIG. 2. Average path length versus network size (N) for the networks investigated in this work (all with average connectivity $\langle k\rangle=4$). Regular rings: clustered $K=2$ (circles), declustered $K=\overline{3}$ (diamonds), and clusters of the square lattice (squares). Complex: small worlds with a probability of rewiring of $p=10\%$—clustered $K=2$ (triangles up), declustered $K=\overline{3}$ (triangles down); scale-free networks with $m_0=5$ and $m=2$ (stars). Lines are fits of the numerical results in the range $N=10-400$: clustered ring $L=0.13N+0.39$, declustered ring $L=0.083N+0.77$, cluster of the square lattice $L=0.59N^{0.47}$, clustered small world $L=1.61\ln N-2.24$, declustered small world $L=1.48\ln N-2.08$, and scale-free network $L=2.16\ln\ln N-0.32$.

III. CHARACTERIZATION OF UNWEIGHTED NETWORKS

A. Average path length and clustering coefficient

Figure 2 shows the average path length L for different types of networks with a links-to-size ratio $N_l/N=2$, versus the size of the system N. Fittings of the numerical results are given in the caption of the figure. In ordered rings, L scales linearly with size $(L_{\rm ring}\sim N)$, with a slope that is smaller for the declustered network $K=\overline{3}$ (the slopes given in the figure caption are close to the exact results—namely, 1/8 and 1/12, respectively; see [42]). For square lattices, the dependence is sublinear—i.e., $L_{\rm square}\approx\sqrt{N}/2$ [42]. Random graphs, in their turn, are known to obey a logarithmic scaling $(L_{\rm rand}\sim\ln N)$ [13]. Such a behavior is also observed in the case of Watts-Strogatz small worlds. As Fig. 2 clearly shows declustered small worlds behave qualitatively in the same way as Watts-Strogatz networks; in both cases the average path length is proportional to $\ln N$. The average path length in declustered small worlds is shorter than in the standard small worlds, as edges of basic square cycles of the initial declustered network couple more distant sites. Finally, scale-free systems appear to be ultrasmall [44], with a double-logarithmic scaling $L_{sf}\propto\ln\ln N$ (see Fig. 2).

In order to differentiate between random graphs and small worlds, both having the same scaling of the average path

TABLE I. Average path length, clustering coefficient, and global and local efficiencies for homogeneous networks.

	L	C	E^g	E_1^l
Regular clustered	12.88	0.5	0.154	0.722
Regular declustered	9.09	0	0.188	0.458
2D square	5.05	0	0.258	0.417
Random	3.40	0.02	0.328	0.280

length and a Poisson distribution of degrees, the clustering coefficient is used. Switching from highly clustered regular graphs to small worlds by introducing a few shortcuts does not significantly alter the clustering coefficient [8,13]. It remains quite large up to high values of the rewiring parameter p. For $p \approx 1$ most triangle loops are broken, leading to random graphs with negligible values of C. The small-world behavior shows up at small p, when both the average path length and the clustering coefficient have large values [8,13].

B. Efficiencies

The aforementioned criteria identify declustered small worlds as random networks. But is this actually the case? Applying the concept of global and the redefined local efficiency, we clearly identify the crucial differences between various networks. The clustered regular ring lattice with $K=2$ is locally very efficient $E_1^l=0.722$, due to its high clusterization; see Table I. Global efficiency is quite low $E^g(N=100)=0.154$, which corresponds to a long average path length $L(100)=12.88$. From our standpoint, a declustered ring lattice with $K=\overline{3}$ has the same characteristics. Local efficiency $E_1^l=0.458$ is relatively good, although the clustering coefficient and the originally proposed local measure of efficiency [36] are both zero. Globally, we obtain slightly a larger value of $E^g(100)=0.188$ [or shorter $L(100)=9.09$], due to the presence of longer-range links. Therefore, regular rings are in general locally efficient and globally inefficient.

Introducing a small number of shortcuts into a regular graph to produce a small-world network does not significantly alter its local topology and local efficiency. On the other hand, global efficiency is appreciably improved. As illustrated in Figs. 3 and 4 this is valid for both clustered and declustered small worlds. The rewiring parameter p was varied in the range 0.01–0.5 (for each value of p results for five graph realizations are shown). In a random graph (large p), the efficiency on the global scale becomes even better, but local efficiency is strongly deteriorated. The distinction between regular (left end), small-world (middle part), and random networks (right end of curves) is clearly depicted in Fig. 5. We can conclude that small worlds are both *globally* and *locally efficient* [36].

Normalizing the global efficiency of a given small world to the values of the initial ring (see Fig. 6), we note that it is improved in a relatively better way in the case of clustered networks. This is due to the fact that initially there are only short $K=2$ links to be rewired into links of longer range. Starting from a declustered regular ring $K=\overline{3}$, links can even-

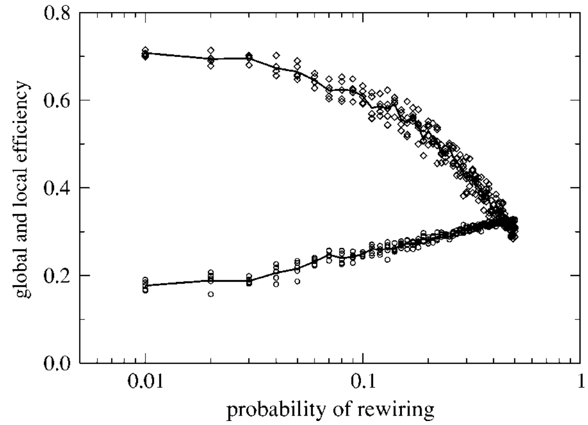

FIG. 3. Global (circles) and local (diamonds) efficiencies versus the rewiring parameter p for clustered small worlds. The initial regular ring was clustered with $K=2$.

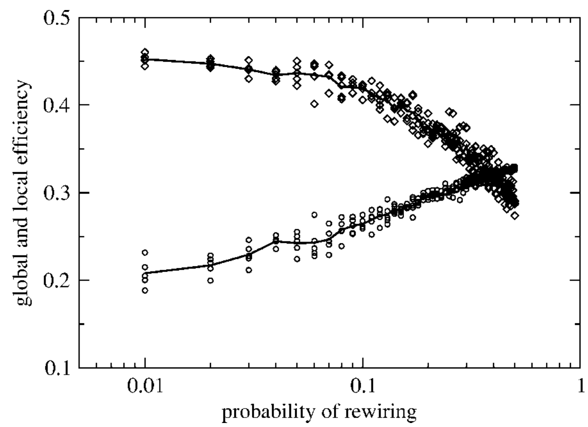

FIG. 4. Global (circles) and local (diamonds) efficiencies versus the rewiring parameter p for declustered small worlds. The initial regular ring was declustered with $K=\overline{3}$.

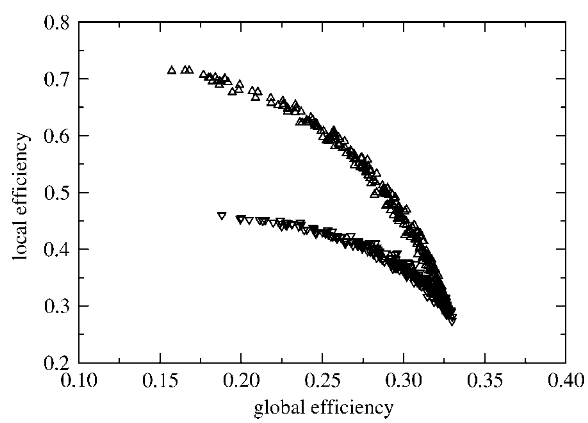

FIG. 5. Global versus local efficiencies for clustered (triangles up) and declustered (triangles down) small worlds. The initial regular rings were clustered ($K=2$, $E^g=0.258$, and $E^l=0.722$) and declustered ($K=\overline{3}$, $E^g=0.188$, and $E^l=0.458$). A rewiring parameter of 0.5 leads to a single random graph.

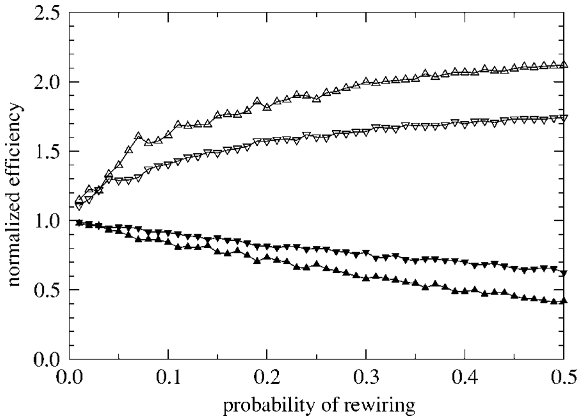

FIG. 6. Global (open symbols) and local (solid symbols) efficiencies normalized to the value of the initial regular ring versus the rewiring parameter p, for clustered (triangles up) and declustered (triangles down) small worlds.

tually be rewired into shorter $K=2$ links, which do not increase the global efficiency. The same type of normalization can be done for the local efficiencies as shown in Fig. 6. Now, possible rewiring of $K=\overline{3}$ links into $K=2$ links actually improves local efficiency, assuring that normalized values for declustered small worlds are always larger than values for the clustered network.

C. Size of the network

The dependence of global and local efficiencies on network size (see Figs. 7 and 8) shows that regular networks have a local efficiency that does vary with N (i.e., unchanged local topology). The global efficiency decreases with size, with the minimal $\epsilon_{ij}^{min}=1/d_{ij}^{max}$ scaling as $1/N$. In the case of small worlds, the local topology is not much affected by the presence of a few shortcuts, leading again to a approximately constant local efficiency. Global efficiency is now expected to decrease at a lower rate, because the minimal $\epsilon_{ij}^{min}=1/d_{ij}^{max}$ scales as $1/\ln N$. The results are quite different for

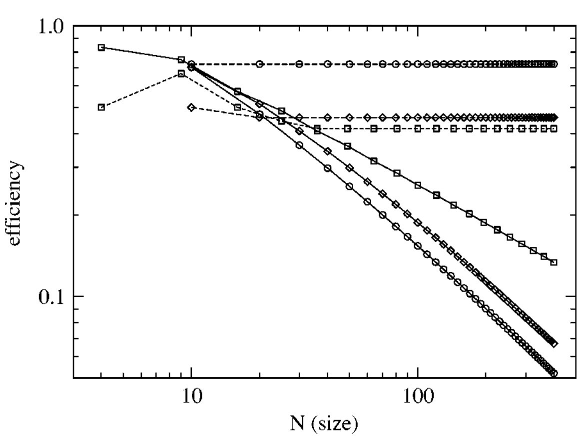

FIG. 7. Global (solid line) and local (dashed line) efficiencies versus network size (N) for regular networks with a constant connectivity ($k_i=4$): clustered $K=2$ (circles), declustered $K=\overline{3}$ (diamonds), and clusters of the square lattice (squares).

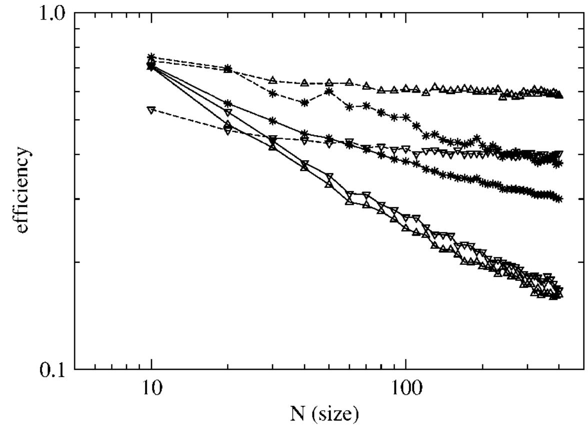

FIG. 8. Global (solid line) and local (dashed line) efficiencies versus the network size (N) of complex networks with a constant average connectivity ($\langle k\rangle=4$). Small worlds with a rewiring parameter $p=0.1$: clustered $K=2$ (triangles up) and declustered $K=\overline{3}$ (triangles down). Scale-free networks with $m_0=5$ and $m=2$ (star).

scale-free networks. While efficiency on a global level decreases with a slower rate with respect to the other networks, local efficiency is significantly decreased. The reason is that the clustering coefficient, giving the main contribution to the local efficiency, decreases exponentially with the size of the ordinary scale-free system [13]. We expect that the local efficiency of highly clustered scale-free networks [35] would be mainly independent of the network size, as their clustering coefficient approaches a high stationary value already for $N\sim 10^2$. Such a tendency is observed in scale-free Internet networks [45], where the clustering coefficient even increases over consecutive years. From our standpoint, the Internet grows, keeping constant local efficiency.

The reason for the observed decrease of global efficiency is related to the fact that, for a constant ratio $N_l/N=a$, increasing the size produces gradually more sparse graphs with longer average path length. The total number of possible links is given by $N(N-1)/2$, while the number of links actually present in the system is $N_l=aN$. This leads to a decrease of the density of links as $\eta=2a/(N-1)$.

D. Normalized global efficiency and basic network

We can normalize the results for each type of network to the values for clustered regular $K=2$ rings of the same size $e^g=E^g/E_r^g$. The results are reported in Fig. 9. The normalized global efficiency of a declustered regular network is slightly larger, but does not change with size. On the contrary, it increases with size for the square lattice, small-world, and scale-free networks. This normalization is necessary if we want to examine how a pure change of topology improves transfer of information, without addition of new links. Equation (3) tells us how efficient is a network on a global scale relatively to the ideal case of a fully connected graph. Such a comparison can be misleading, because does not take into account that graphs are commonly sparse. Increasing the size, while keeping the N_l/N ratio constant, global efficiency decreases for any kind of sparse networks. In contrast to a

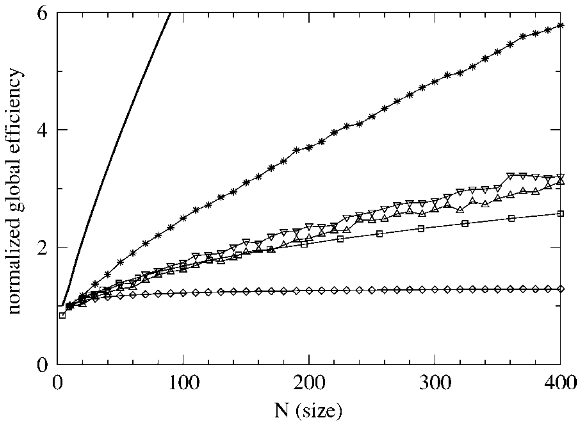

FIG. 9. Normalized global efficiency versus network size (N) with a constant average connectivity ($\langle k\rangle=4$). The global efficiency of each particular network is normalized to the value for the regular clustered $K=2$ network of the same size. Regular: declustered $K=\overline{3}$ (diamonds) and clusters of the square lattice (squares). Complex: small worlds with a rewiring parameter $p=0.1$—clustered $K=2$ (triangles up) and declustered $K=\overline{3}$ (triangles down); scale-free networks with $m_0=5$ and $m=2$ (stars). Fully connected graph: solid line.

fully connected graph, each type of network would be seen as inefficient, no matter what is the underlying topology. Therefore our opinion is that a particular network should be compared with a corresponding basic network with the same number of sites N and links N_l. The basic network is a periodic system with the longest possible average path length or the smallest possible global efficiency for given (N,N_l). It can be constructed in the following way.

(a) Start from an initial standard ring of N sites and N links.

(b) Add links between each site and its closest surrounding sites (the next-nearest neighbors, the next-next-nearest neighbors, and so on), up to all N_l links are used. The result is a $K=N_l/N$ regular ring.

(c) In case that the ratio N_l/N is not an integer, the last set of $n_l<N$ links should be evenly distributed among the sites.

In such a way, complex systems such as small worlds or scale-free networks are identified to be globally efficient in comparison with the corresponding inefficient basic (regular) networks.

IV. WEIGHTED NETWORKS

A. Efficiency measures

In this section we focus on a particular type of weighted graphs, where physical distances are introduced. A real network is described by both the connectivity matrix and the matrix of physical distances [36]. The shortest physical path length $\tilde{d}_{ij}$ between two sites i and j is the path with the smallest sum of distances, no matter the number of links the network has. Only in the case of links of equal lengths (λ) do the physical and graph shortest paths coincide—i.e., $\tilde{d}_{ij}=\lambda d_{ij}$. The efficiencies of a real network could be calculated using the formulas given in Sec. II C, replacing d_{ij} by $\tilde{d}_{ij}$. In order to keep these quantities dimensionless, a suitable normalization should be performed. The originally proposed efficiency measures [36] are normalized to the values for the fully connected graph of the same size:

$$\tilde{E}_1^g=\frac{\sum_{i\neq j}\frac{1}{\tilde{d}_{ij}}}{\sum_{i\neq j}\frac{1}{l_{ij}}},\tag{6}$$

where l_{ij} is a physical distance (or length of a possible direct link) between sites i and j. We propose a slightly different measure, which gives similar quantitative results. Instead of comparing the network as a whole with the ideal graph, we analyze the efficiency of each particular shortest path $\tilde{d}_{ij}$ separately:

$$\tilde{E}_2^g=\frac{1}{N(N-1)}\sum_{i\neq j}\frac{l_{ij}}{\tilde{d}_{ij}}.\tag{7}$$

The main reason for using this network average of path efficiencies is that in weighted networks the shortest paths going through a lot of sites can very often be as efficient as direct links between pairs of sites, significantly contributing to the network efficiency. If we neglect possible delays between received and subsequent emitted information, we see that the straight path going through many sites is the same as a direct straight link between two end nodes. The weighted efficiency of such a straight-path regular graph will be the same as that of the corresponding fully connected system. This simple example raises a question: is it necessary at all to impose a small-world topology in order to achieve a higher global efficiency in a weighted network? A closer look into a $K=1$ weighted ring shows that the weighted efficiency of the longest path between two opposite sites is very high—i.e., $l_{ij}/\tilde{d}_{ij}\approx 2R/(R\pi)=2/\pi$. That is, 64% of the efficiency of the direct link. For sites closer to each other or for a regular network with $K>1$, this ratio is even larger. This result is not surprising, as we assume that the speed of informational transfer through all the links is constant. Therefore links between faraway sites do not represent shortcuts, because the time needed to transfer the information increases with the physical length of the link. The shortcut would be created if the transfer is instantaneous or at least very fast, so that the corresponding transfer time is much shorter than the characteristic times of the underlying dynamics. Such a case would be a flight of an infected person by an airplane, when the basic mechanism is a slow spreading of the disease through direct contacts [4], but definitely not the transportation systems (like railways) where the speed of all the vehicles is limited and usually constant.

Similar measures can be defined on the local scale. The original weighted local efficiency, when paths are going only through the first nearest neighbors of a reference site, is given by [36]

$$\tilde{E}_0^l = \frac{\sum_i \frac{1}{k_i[k_i-1]} \sum_{j\neq k\in\Gamma_1} \frac{1}{\tilde{d}^0_{jk/i}}}{\sum_i \frac{1}{k_i[k_i-1]} \sum_{j\neq k\in\Gamma_1} \frac{1}{l_{jk}}}. \tag{8}$$

Allowing for paths going through the rest of the network we define $\tilde{E}_1^l$ by simply replacing $\tilde{d}^0_{jk/i}$ by $\tilde{d}_{jk/i}$. Finally, we can again normalize each path separately, instead of normalizing the whole sum by the value for fully connected network ($\tilde{E}_2^l$).

B. Analysis of subway transportation systems

Underground transportation networks are important complex (but not random) systems with negligible clustering coefficient. Despite being small in size, they are ideal examples for demonstrating the strength of our method for the analysis of local efficiency. We have made a reinterpretation of the results obtained for the Boston subway network [36] and performed an analysis of Barcelona (B) and Madrid (M) underground systems.

The Boston underground transportation system, consisting of $N=124$ stations and $N_l=124$ tunnels, was described both as an unweighted and a weighted graph in [36]. In the unweighted case, it was found that it is neither globally nor locally efficient, having $E^g=0.1$ and $E_0^l=0.006$. This small value for E^g gives a false impression of low global efficiency. Although it is only 10% of the largest value for fully connected graph, we should check how much the complex topology of the Boston subway system improves its efficiency, compared to a regular ring with the same number of sites and links. We found out that such a ring has $E_r^g=0.076$, so that the Boston network is by 32% more efficient. Locally, the original measure relying heavily on the presence of triangles of neighbors has a very small value $E_0^l=0.006$ [36], as a consequence of the typically low clustering in underground transportation systems. Another comparison could be made against a hub consisting of a central node of degree $k_c=125$ and 125 peripheral nodes. Such a graph has the highest possible global efficiency of $E^g_{hub}\approx 0.5$ for the given number of 125 links, but local efficiency (E_0^l or E_1^l) is zero. Any attempt to locally increase efficiency of a hub by rearranging links would eventually lead to a decrease of it on the global scale. Therefore, we consider that in real systems, such as the Boston subway network, an appropriate pay-off between global and local efficiencies is achieved. Taking physical distances into account, the global efficiency is increased to $\tilde{E}_1^g=0.63$, while locally remains quite low $\tilde{E}_0^l=0.03$ [36]. Only after the network is extended to include the Boston bus system does it become efficient on both scales, with $\tilde{E}_1^g=0.72$ and $\tilde{E}_0^l=0.46$ [36]. This final result was interpreted as a small-world behavior. On the basis of our previous discussion of weighted regular networks it is evident that such an interpretation is not correct. A simple weighted regular ring with $K=2$ is both globally and locally very efficient, due to the constant speed of trains and high clustering coefficient, respectively. Weighted efficiencies in real networks with constant speed of informational transfer are not appropriate measures to give clear criteria for its classification. They can only give a hint on up to which extent a particular real network can replace the ideal fully connected weighted graph. Furthermore, it seems that only the comparison with the ideal graph is plausible. In most of the cases it is hard to find out what should be the corresponding weighted "regular" network, because the geographical positions of the nodes in a real complex network are given and fixed, and usually not equidistant.

TABLE II. Performances of Barcelona and Madrid subway systems.

	Barcelona	Madrid
N	104	188
N_l	115	223
L	9.85	12.36
C	0.008	0.011
E^g	0.153	0.127
E_0^l	0.009	0.012
E_1^l	0.080	0.115
$\tilde{E}_1^g$	0.734	—
$\tilde{E}_2^g$	0.753	—
$\tilde{E}_0^l$	0.019	—
$\tilde{E}_1^l$	0.136	—
$\tilde{E}_2^l$	0.131	—

In the following analysis of Barcelona and Madrid subways we will include several technical details and make a step outside a pure theoretical research, offering proposals on how efficiencies of these networks could be improved. Concerning the Barcelona system [46], we do not take into account connections by a regular train, but only consider six metro lines. The number of stations and tunnels are $N(\mathrm{B})=104$ and $N_l(\mathrm{B})=115$, respectively. When viewed as an unweighted graph, this system has the average path length of $L(\mathrm{B})=9.85$, very small clustering coefficient $C(\mathrm{B})=0.008$ (which is the most important contribution to E_0^l), global efficiency of $E^g(\mathrm{B})=0.153$, and a redefined local efficiency of $E_1^l(\mathrm{B})=0.080$ (see Table II). Comparing with the corresponding basic (regular) network with $L_b=23.58$ and $E_b^g=0.095$, we see that the average path length is more than two times shorter and the global efficiency improved by 61%, due to the complex topology of the Barcelona system. Furthermore, the local efficiency is nine to 10 times larger than if it would have been estimated on the basis of the original equation [36] or the clustering coefficient. Similar results are obtained for the Madrid system [47]. It consists of 13 metro lines (including the ring *MetroSur*), forming an unweighted network of $N=188$ nodes and $N_l=223$ links. Due to its larger size, the average path length $L(\mathrm{M})=12.36$ is longer than in the Barcelona system and the global efficiency is smaller $E^g(\mathrm{M})=0.127$; see Table II. Nevertheless, the values of these two quantities are much better than for the corresponding basic network with $L_b=38.64$ and $E_b^g=0.064$. The complex topology improves the global efficiency by more than 98%. The clustering coefficient is larger than in Barcelona [$C(\mathrm{M})=0.011$], because several triangles are formed (particularly around stations *Gran Via* and *Goya*). The higher redefined local efficiency of $E_1^l(\mathrm{M})=0.115$ is a consequence of the larger clustering coefficient, as well as the presence of two rings (metro lines number 6 *Circular* and number 12 *MetroSur*). Cutting off a reference site belonging to one of these two rings, its ring neighbors can still exchange trains along the rest of the ring.

Similarly to the Boston subway system, the global efficiency of the weighted Barcelona network is quite high: $\widetilde{E}_1^g(\mathrm{B})=0.734$ or $\widetilde{E}_2^g(\mathrm{B})=0.754$. The main contributions come from several straight-line subgraphs (such as that between stations *Santa Eulalia* and *Sagrada Familia*), being identical to fully connected weighted subgraphs. The redefined local efficiency takes values of $\widetilde{E}_1^l(\mathrm{B})=0.136$ or $\widetilde{E}_2^l(\mathrm{B})=0.131$, which are about 7 times larger than when calculated using the original equation [36]; see Table II. The efficiencies could be further improved by directly connecting a few stations that are separated by a long path, although physically close to each other. Adding only two links, one between stations *Can Serra* and *Can Vidalet* and another between *Valldaura* and *Horta*, two new rings are created. The number of links is increased by only 1.7%, while the increase of the global efficiency is $\delta\widetilde{E}_1^g(\mathrm{B})=3.3\%$ and that of the local efficiency $\delta\widetilde{E}_1^l(\mathrm{B})=26.5\%$. Obviously, we can even assume that the stations within these pairs are *connected* or represent a *single station*, as we can simply walk from one to the other.

V. CONCLUDING REMARKS

In this work we focused on two objectives: (i) introducing a new definition of local efficiency that does not depend exclusively on the clustering coefficient and (ii) using that definition to show that there is another class of complex networks with short average path length and Poisson distribution of degrees that is not random although its clustering coefficient is negligible.

After accomplishing the first task, we proceeded with a systematic analysis of different types of regular and complex networks. Calculating global and modified local efficiencies and taking into account the distribution of connectivity, we were able to make a clear classification of *unweighted* complex networks. The main conclusions that emerge from this study are the following.

(i) The class of small worlds can be generalized to include systems with negligible clustering coefficient. We introduced a new type of networks that has a small number of triangle cycles, but still clearly distinguishable from random systems due to its relatively good local transfer of information.

(ii) Small worlds (both clustered and declustered) as homogeneous systems and scale-free networks as heterogeneous systems are identified to be both globally and locally efficient.

Showing that declustered small worlds behave qualitatively in the same way as standard clustered small worlds, we addressed today's paradigm of the importance of the clustering coefficient.

Applying our method to real networks with physical distances and a constant speed of informational transfer, we found that *weighted* efficiencies can be used only to compare a particular real network with the ideal fully connected weighted graph. As highly clustered weighted regular rings can be both globally and locally efficient, it is hard to establish clear criteria for identification of small-world behavior. In particular our analysis of the underground transportation systems of Boston [36], Barcelona, and Madrid reveals a proper balance between global and local performance. Despite the constraints on the number of tunnels, global efficiency is noticeably high due to the complex topology of these networks. On the other hand, allowing for the use of alternative paths after one station is cut off, the local efficiency turns to be 5–10 times larger than the results reported in Ref. [36].

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